Probability

1. Probability that a new record will be set up $= 1-\left(1-\frac{1}{4}\right)\left(1-\frac{2}{7}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{2}{5}\right)\left(1-\frac{1}{2}\right)=\underline{\underline{\frac{25}{28}}}$

2. Probability of A to win a game $= 2/3$

Probability of B to win a game $= 1/3$

Pr (A wins at least 3 games in a set of five) = ${}_5C_5\left(\frac{2}{3}\right)^5+{}_5C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)+{}_5C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2=\underline{\underline{\frac{192}{243}}}$

Pr (A wins 5 games before B wins two)

= Pr(A wins 5 games) + Pr(A wins 4 games) + Pr(A wins 3 games in the first 4 games)

$={}_5C_5\left(\frac{2}{3}\right)^5+{}_5C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)+{}_4C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^1=\underline{\underline{\frac{48}{81}}}$

3. **(a)** Pr(2 appears every time) = $\left(\frac{1}{6}\right)^5=\underline{\underline{\frac{1}{7776}}}$

(b) Pr(2 appears exactly 4 times) = ${}_5C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)=\underline{\underline{\frac{25}{7776}}}$

(c) Pr(2 appears at least 3 times) = $\frac{1}{7776}+\frac{25}{7776}+{}_5C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2=\underline{\underline{\frac{276}{7776}}}$

4. The total number of ways of taking any number of balls :

T = C(n,1) + C(n,2) + C(n,3) + … + C(n,n)

The total number of ways of taking even number of balls :

E= C(n,2) + C(n,4) + C(n,6) + … + C(n,[n/2] × 2)

where [a] is defined as the biggest integer less than or equal to a.

Since $(1+x)^n=\sum_{i=0}^{n}C(n,i)x_i$,

Putting n = 1, $2^n=(1+1)^n=\sum_{i=0}^{n}C(n,i)$ …..(1)

$\therefore$ T = $2^n - 1$

Putting n = –1, $0^n=(1-1)^n=\sum_{i=0}^{n}(-1)^nC(n,i)$ …..(2)

{(1) + (2)}/2, $\therefore$ E = $2^{n-1} - 1$

Pr(an even number of balls are taken) $=\frac{E}{T}=\underline{\underline{\frac{2^{n-1}-1}{2^n-1}}}$

5. The probability of obtaining 1 in not more than n trials

$=\frac{1}{6}+\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)+\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2+\ldots+\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1}=1-\left(\frac{5}{6}\right)^n>\frac{1}{2}\Rightarrow n>\log\frac{1}{2}\Big/\log\frac{5}{6}\approx 3.8$

Therefore the number of times the die must be thrown = $\underline{\underline{4}}$

6. Pr(A obtains 1 first) = Pr(B obtains 1 first | A loses 1st game)

= Pr (B obtains 1 first $\cap$ A loses 1st game) / Pr (A loses 1st game)

= Pr(B obtains 1 first) / [5/6] …..(1)

But Pr(A obtains 1 first) + Pr(B obtains 1 first) = 1 …..(2)

Solving (1) and (2) , Pr(A obtains 1 first) = $\underline{\underline{\dfrac{6}{11}}}$

Pr(B obtains 1 first) = $\underline{\underline{\dfrac{5}{11}}}$

7. If the balls drawn cannot be replaced, Pr(A draws the white ball first) = $\dfrac{3}{5}+\left(\dfrac{2}{5}\right)\left(\dfrac{1}{4}\right)=\underline{\underline{\dfrac{7}{10}}}$

Pr(B draws the white ball first) = $1-\dfrac{7}{10}=\underline{\underline{\dfrac{3}{10}}}$

If the balls drawn can be replaced,

Pr(A draws the white ball first) = $\dfrac{3}{5}+\left(\dfrac{2}{5}\right)\left(\dfrac{2}{5}\right)\left(\dfrac{3}{5}\right)+\left(\dfrac{2}{5}\right)\left(\dfrac{2}{5}\right)\left(\dfrac{2}{5}\right)\left(\dfrac{2}{5}\right)\left(\dfrac{3}{5}\right)+....(\text{Infinite G.P.})=\underline{\underline{\dfrac{5}{7}}}$

Pr(B draws the white ball first) = $1-\dfrac{5}{7}=\underline{\underline{\dfrac{2}{7}}}$

8. Pr(the ball is white)

= Pr(the ball is white $\cap$ urn A is chosen) + Pr(the ball is white $\cap$ urn B is chosen)

= Pr(urn A is chosen) Pr(the ball is white | urn A) + Pr(urn B is chosen) Pr(the ball is white | urn B)

= $\left(\dfrac{1}{2}\right)\left(\dfrac{3}{10}\right)+\left(\dfrac{1}{2}\right)\left(\dfrac{4}{12}\right)=\underline{\underline{\dfrac{19}{60}}}$

9. **(a)** The sample space, S , is {(1,1), (1,2), …(1,6), (2,1),(2,2),…,(2,6),…..(6,1),(6,2),…,(6,6)}

N(S) = 36

The event space, E_1 , is {(3,6),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6)} $\therefore$ $N(E_1) = 10$

Pr(the sum is at least 9) = $\dfrac{N(E_1)}{N(S)}=\dfrac{10}{36}=\underline{\underline{\dfrac{5}{18}}}$

(b) The event space, E_2 , is

{(1.2),(2,1), (2,3),(3,2),(3,4),(4,3),(4,5),(5,4),(5,6),(6,5),(1,3),(3,1),(2,4),(4,2),(3,5),(5,3),(4,6),(6,4)}

N(E) = 18

Pr(the difference of the numbers in two dices is 1 or 2) = $\dfrac{N(E_2)}{N(S)}=\dfrac{18}{36}=\underline{\underline{\dfrac{1}{2}}}$

10. The probability = $\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)\left(\dfrac{1}{6}\right)=\underline{\underline{\dfrac{1}{24}}}$

11. The probability = ${}_5C_1\left(\frac{1}{4}\right)^4\left(\frac{1}{2}\right)=\underline{\underline{\frac{5}{32}}}$

12. If the last defective is on the 7^{th} testing, the first defective must be among the first six testing.

Therefore the number of ways in which the last defective is on the 7^{th} testing $= 6$

Total number of arrangements in which the bad light bulbs is on any testing = ${}_{10}C_2 = 45$

The probability = $\frac{6}{45}=\underline{\underline{\frac{2}{15}}}$

13. The total number of ways in choosing 5 prizes from n tickets = ${}_nC_5$.

The number of ways in choosing 5 prizes in which the 2 tickets are not prizes = ${}_{n-2}C_5$.

Therefore Pr(win at least one prize) = 1 – Pr(win no prize) $=1-\frac{{}_{n-2}C_5}{{}_nC_5}=\underline{\underline{\frac{10n-30}{n^2-n}}}$

14. Let the tossing be denoted by (X|Y).

Total number in the sample space = $2^4 = 16$

Event space in which $Y – X$ will be less than 1 = {(HH|HH),(HT|HT),(TH|TH),(HT|TH),
(TH|HT),(TT|TT),(HH|TT),(HT|TT),(TH|TT),(HH|TH),(HH|HT) }

There are 11 number of elements in the event space.

Therefore the probability that $Y – X$ will be less than 1 = $\underline{\underline{\frac{11}{16}}}$

<u>Conditional Probability</u>

15. Let M = the person is a man, W = the person is a woman, X = the person has a ticket

P(M) = 0.6, P(W) = 0.4, P(X|M) = 0.8, P(X|W) = 0.75

By Bayes' Theorem,

$$P(M\mid X)=\frac{P(M)P(X\mid M)}{P(M)P(X\mid M)+P(W)P(X\mid W)}=\frac{0.6\times 0.8}{0.6\times 0.8+0.4\times 0.75}=\underline{\underline{\frac{8}{13}}}$$

$$P(W|X) = 1-\frac{8}{13}=\underline{\underline{\frac{5}{13}}}$$

16. Let A = the die falls "1" or "2" = 1^{st} urn is chosen

B = the die falls "3" = 2^{nd} urn is chosen

P(A) = 2/3, P(B) = 1/3, P(X|A) = 1/5, P(X|B) = 3/5

By Bayes' Theorem,

$$P(A\mid X)=\frac{P(A)P(X\mid A)}{P(A)P(X\mid A)+P(B)P(X\mid B)}=\frac{(2/3)(1/5)}{(2/3)(1/5)+(1/3)(3/5)}=\underline{\underline{\frac{2}{5}}}$$

17. **(a)** Let B = 5 cards are black

A = at least 4 cards are black

$$P(B) = \frac{{}_{26}C_5}{{}_{52}C_5} \qquad P(A) = \frac{{}_{26}C_5 + {}_{26}C_4 \times {}_{26}C_1}{{}_{52}C_5}, \qquad P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{P(A)} = \frac{11}{76}$$

(b) Let B = 5 cards are black

A = at least 4 spades

$$P(B) = \frac{{}_{26}C_5}{{}_{52}C_5} \qquad P(A) = \frac{{}_{13}C_5 + {}_{13}C_4 \times {}_{13}C_1}{{}_{52}C_5}, \; P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{P(A)} = \frac{37}{102}$$

18. Let A_i = Box i is chosen

X = a gold coin is chosen

$P(A_i) = 1/3$, $P(X|A_1) = 1$, $P(X|A_2) = 0$, $P(X|A_3) = 1/2$,

By Bayes' Theorem, $$P(A_1 \mid X) = \frac{P(A_1)P(X \mid A_1)}{\sum_{i=1}^{3} P(A_i)P(X \mid A_i)} = \frac{(1)(1/3)}{(1)(1/3) + (0)(1/3) + (1/2)(1/3)} = \frac{2}{3}$$

19. Let A = the first ball is white

B = the second ball is white

$P(A \cap B) = P(A)P(B|A) = (2/4)(1/4) = 1/8$

$P(B) = P(A)P(B|A) + P(A')P(B|A') = (2/4)(1/4) + (2/4)(3/4) = 1/2$

Therefore, $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/8}{1/2} = \frac{1}{4}$$

20. Two dices are rolled and one or more of the faces is 6.

The sample space, S = { (1,6),(2,6),(3,6),(4,6),(5,6),(6,6),(6,1),(6,2),(6,3),(6,4),(6,5) }

The event space in which the sum of the faces exceeds 8 is:

E = = { (3,6),(4,6),(5,6),(6,6), (6,3),(6,4),(6,5) }

Therefore the probability = $\frac{N(E)}{N(S)} = \frac{7}{11}$

21. Let K = the card is King

F = the card is a Face card

$$P(K \mid F) = \frac{P(K)}{P(F)} = \frac{4}{12} = \frac{1}{3}$$